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Synchronization in systems with multiple time delays

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We report on chaos synchronization between two unidirectionally coupled chaotic systems with multiple time delays and find both the existence and stability conditions for anticipating, lag, inverse and complete synchronizations. The method is tested on the famous Ikeda model. Numerical simulations fully support the analytical approach.

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I. INTRODUCTION

Seminal papers on chaos synchronization [1] have stimulated a wide range of research activity [2]. Synchronization phenomena in coupled systems have been especially extensively studied in the context of laser dynamics, electronic circuits, chemical and biological systems [2]. Application of chaos synchronization can be found in secure communication, optimization of nonlinear system performance, modeling brain activity and pattern recognition phenomena [2].

Due to finite signal transmission times, switching speeds and memory effects systems with both single and multiple delays are ubiquitous in nature and technology [3]. Dynamics of multifeedback systems are representative examples of the multidelay systems. Therefore the study of synchronization phenomena in time-delayed systems is of high practical importance. Prominent examples of such dynamics can be found in biological and biomedical systems, laser physics, integrated communications [3]. In laser physics such a situation arises in lasers subject to two or more optical or electro-optical feedback. Second optical feedback could be useful to stabilize laser intensity [4]. Chaotic behavior of laser systems with two optical feedback mechanisms is studied in recent works [5]. To the best of our knowledge, chaos synchronization between the multifeedback systems is yet to be investigated. Having in mind enormous application implications of chaos synchronization, e.g., in secure communication, investigation of synchronization regimes in the multifeedback systems is of certain importance.

Recently there have been several reports on synchronization in the systems with multiple delays. In [6] the authors studied unidirectionally coupled discrete systems; papers [7,8] deal with bidirectionally coupled multiple-delay systems.

In this paper we investigate synchronization between two unidirectionally coupled continuous chaotic systems with mutiple time delays and find both the existence and stability conditions for different synchronization regimes. We test the approach on the paradigm Ikeda model. We support the analytical approach with numerical simulations.

II. GENERAL APPROACH

Consider synchronization between the double-feedback systems of general form,

$$\frac{dx}{dt} = -\alpha x + m_1 f(x_{\tau_1}) + m_2 f(x_{\tau_2}), \tag{1}$$

$$\frac{dy}{dt} = -\alpha y + m_3 f(y_{\tau_1}) + m_4 f(y_{\tau_2}) + K f(x_{\tau_3}), \tag{2}$$

where f is the differentiable generic nonlinear function. Throughout this paper $x_\tau \equiv x(t-\tau)$. One finds that under the condition

$$K = m_1 - m_3, m_2 = m_4, \tag{3}$$

Eqs. (1) and (2) admit the synchronization manifold

$$y = x_{\tau_3-\tau_1}. \tag{4}$$

This follows from the dynamics of the error $\Delta = x_{\tau_3-\tau_1} - y$

$$\frac{d\Delta}{dt} = -\alpha\Delta + m_3 \Delta_{\tau_1} f'(x_{\tau_3}) + m_2 \Delta_{\tau_2} f'(x_{\tau_2+\tau_3-\tau_1}). \tag{5}$$

Here f' stands for the derivative of f with respect to time and the derivative should be bounded. The sufficient stability condition of the trivial solition $\Delta=0$ of Eq. (5) can be found from the Krasovskii-Lyapunov functional approach ([3], p. 154) (see also seminal paper [9] on the first application of the Krasovskii-Lyapunov functional to chaos synchronization in time-delayed systems). According to [3], the sufficient stability condition for the trivial solution $\Delta=0$ of time-delayed equation $d\Delta/dt = -r(t)\Delta + s_1(t)\Delta_{\tau_1} + s_2(t)\Delta_{\tau_2}$ is $r(t) > |s_1(t)| + |s_2(t)|$. Thus we obtain that the sufficient stability condition for the synchronization manifold $y = x_{\tau_3-\tau_1}$ (4) can be written as

$$\alpha > |m_3(\sup f'(x_{\tau_3}))| + |m_2(\sup f'(x_{\tau_2+\tau_3-\tau_1}))|. \tag{6}$$

Here $\sup f'(x)$ stands for the supremum of f' with respect to the appropriate norm.

Analogously one finds both the existence ($m_2 - K = m_4$, $m_1 = m_3$) and sufficient stability $[\alpha > |m_3(\sup f'(x_{\tau_3}))| + |m_2(\sup f'(x_{\tau_2+\tau_3-\tau_1}))|]$ conditions for synchronization manifold $y = x_{\tau_3-\tau_2}$. One can also find the existence ($m_3 = m_1 + K$, $m_2 = m_4$) and sufficient stability $[\alpha > |m_3(\sup f'(x_{\tau_3}))| + |m_2 \sup(f'(x_{\tau_2+\tau_3-\tau_1}))|]$ conditions for the inverse synchronization [10] manifold $y = -x_{\tau_3-\tau_1}$ (we notice that this result is valid if f is an odd function of x). Further generalization of the approach to n-tuple feedback systems, i.e., systems with

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multiple delays of type (1) and (2) is straightforward. We underline that a stability condition derived from the Krasovskii-Lyapunov approach is a sufficient condition: it assures a high quality synchronization for a coupling strength estimated from the stability condition, but does not forbid the possibility of synchronization with smaller coupling strengths. The threshold coupling strength can be estimated by the dependence of the maximal Lyapunov exponent λ of the error dynamics on K, i.e., from $\lambda(K)=0$ [9].

III. CHAOS SYNCHRONIZATION BETWEEN THE IKEDA SYSTEMS WITH MULTIPLE DELAYS

In this section of the paper we test the approach presented in Sec. II on the Ikeda model-paradigm model in chaotic dynamics. Consider synchronization between the multifeedback Ikeda systems,

$$\frac{dx}{dt} = -\alpha x + m_1 \sin x_{\tau_1} + m_2 \sin x_{\tau_2}, \tag{7}$$

$$\frac{dy}{dt} = -\alpha y + m_3 \sin y_{\tau_1} + m_4 \sin y_{\tau_2} + K \sin x_{\tau_3}, \tag{8}$$

with positive $\alpha_{1,2}$ and $-m_{1,2,3,4}$.

This investigation is of considerable practical importance, as the equations of the class B lasers with feedback (typical representatives of class B are solid-state, semiconductor, and low pressure CO_2 lasers [11]) can be reduced to an equation of the Ikeda type [12].

The Ikeda model was introduced to describe the dynamics of an optical bistable resonator, plays an important role in electronics and physiological studies and is well-known for delay-induced chaotic behavior [13,14]; see also, e.g. [10,15]. Physically x is the phase lag of the electric field across the resonator; α is the relaxation coefficient for the driving x and driven y dynamical variables; $m_{1,2}$ and $m_{3,4}$ are the laser intensities injected into the driving and driven systems, respectively. $\tau_{1,2}$ are the feedback delay times in the coupled systems; τ_3 is the coupling delay time between systems x and y; K is the coupling rate between the driver x and the response system y.

We establish that systems (7) and (8) can be synchronized on

$$y = x_{\tau_3-\tau_1} \tag{9}$$

as the error signal $\Delta = x_{\tau_3-\tau_1} - y$ for small Δ under the condition

$$K = m_1 - m_3, m_2 = m_4 \tag{10}$$

obey the dynamics

$$\frac{d\Delta}{dt} = -\alpha\Delta + m_3\Delta_{\tau_1} \cos x_{\tau_3} + m_2\Delta_{\tau_2} \cos x_{\tau_2+\tau_3-\tau_1}. \tag{11}$$

It is obvious that $\Delta=0$ is a solution of system (11). We notice that for $\tau_3>\tau_1$, $\tau_3=\tau_1$, and $\tau_3<\tau_1$ (9) is the retarded, complete, and anticipating synchronization manifold [14,10,15], respectively.

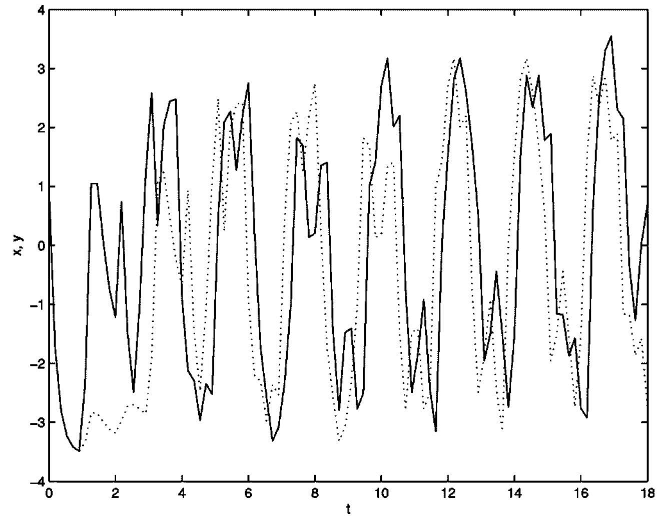

FIG. 1. Numerical simulation of the Ikeda model, Eqs. (7) and (8): the time series of the driver $x(t)$ (solid line) and the driven system $y(t)$ (dotted line) for $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=3$, $m_1=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-17$. Dimensionless units.

By using the Krasovskii-Lyapunov functional approach we obtain that the sufficient stability condition for the synchronization manifold $y=x_{\tau_3-\tau_1}$ can be written as

$$\alpha > |m_3| + |m_2|. \tag{12}$$

As Eq. (11) is valid for small Δ stability condition (12) found above holds locally. Conditions (10) are the existence conditions for the synchronization manifold (9) between unidirectionally coupled Ikeda systems (7) and (8) with multiple delays.

We would like to emphasize that conditions (10) and (12) can be satisfied easily, as the number of parameters exceeds the number of restrictions. In the cases of parameter mismatches (e.g., $m_2 \neq m_4$, $m_1-K\neq m_3$, the feedback delay times are different for the driver and driven systems, etc.), i.e., in the study of nonidentical coupled systems generalized synchronization [16] between the driver and driven systems is observed under sufficiently strong driving, when there is some functional relation between the states of response and drive, i.e., $y(t)=F(x(t))$. One can use the auxiliary system method to detect generalized synchronization: that is given another identical driven auxiliary system z, generalized synchronization between x and y is established with the achievement of complete synchronization between y and z. Investigation of generalized synchronization in systems with multiple time delays is under progress and will be presented elsewhere.

By investigating corresponding error dynamics we find that $y=x_{\tau_3-\tau_2}$ is the synchronization manifold between systems (7) and (8) with the existence $m_2-K=m_4$ and $m_1=m_3$ and stability conditions $\alpha>|m_3|+|m_4|$.

One can generalize the previous results to n-tuple feedback Ikeda systems. Applying the error dynamics approach to synchronization between the following Ikeda models

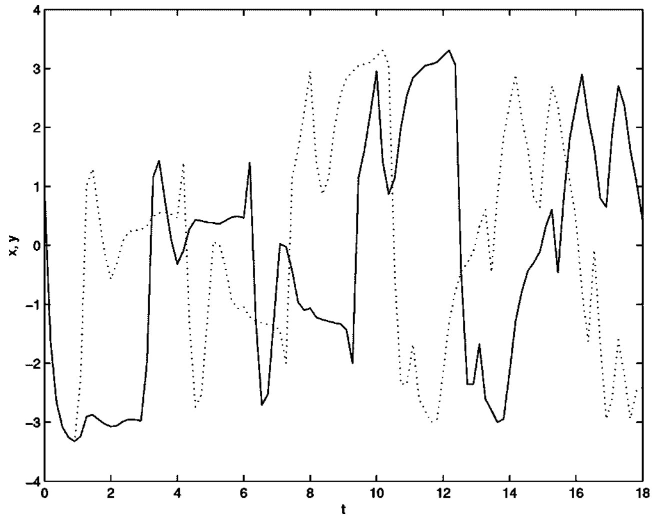

FIG. 2. Numerical simulation of the Ikeda model, Eqs. (7) and (8): the time series of the driver $x(t)$ (solid line) and the driven system $y(t)$ (dotted line) for $\alpha=5$, $\tau_1=2$, $\tau_2=3$, $\tau_3=1$, $m_1=m_3=-2$, $m_2=-18$, $m_4=-1$, and $K=-17$. Dimensionless units.

$$\frac{dx}{dt} = -\alpha x + m_{1x} \sin x_{\tau_1} + m_{2x} \sin x_{\tau_2} + \cdots + m_{nx} \sin x_{\tau_n}, \tag{13}$$

$$\frac{dy}{dt} = -\alpha y + m_{1y} \sin y_{\tau_1} + m_{2y} \sin y_{\tau_2} + \cdots + m_{ny} \sin y_{\tau_n} + K \sin x_{\tau_k}, \tag{14}$$

we find that the existence and sufficent stability conditions, e.g., for the synchronization manifold $y=x_{\tau_k-\tau_1}$ are: $m_{1x}-K=m_{1y}$, $m_{nx}=m_{ny}$ and $\alpha>|m_{1y}|+|m_{2y}|+\cdots+|m_{ny}|$, respectively. For the synchronization manifold $y=x_{\tau_k-\tau_2}$, $m_{2x}-K=m_{2y}$ and $m_{nx}=m_{ny}$ are the existence conditions, and $\alpha>|m_{1y}|+|m_{2y}|+\cdots+|m_{ny}|$ is the sufficient stability condition.

Numerical simulations fully support the analytical results. Equations (7) and (8) were simulated using the DDE23 program [17] in MATLAB 6. Figure 1 shows the time series of the driver $x(t)$ (solid line) and the driven system $y(t)$ (dotted line) for $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=3$, $m_1=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-17$. After transients the driven system shifted $\tau_3-\tau_1=2$ time units to the right and $y=x(t-2)$ (lag synchronization). In Fig. 2 the time series of the driver $x(t)$ (solid line) and the driven system $y(t)$ (dotted line) for $\alpha=5$, $\tau_1=2$, $\tau_2=3$, $\tau_3=1$, $m_1=m_3=-2$, $m_2=-18$, $m_4=-1$ and $K=-17$. After transients the driven system shifted $\tau_3-\tau_2=-2$ time units to the left and $y=x(t+2)$ (anticipating synchronization). Figure 3 shows complete synchronization between x and y for the parameters $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=1$, m_1

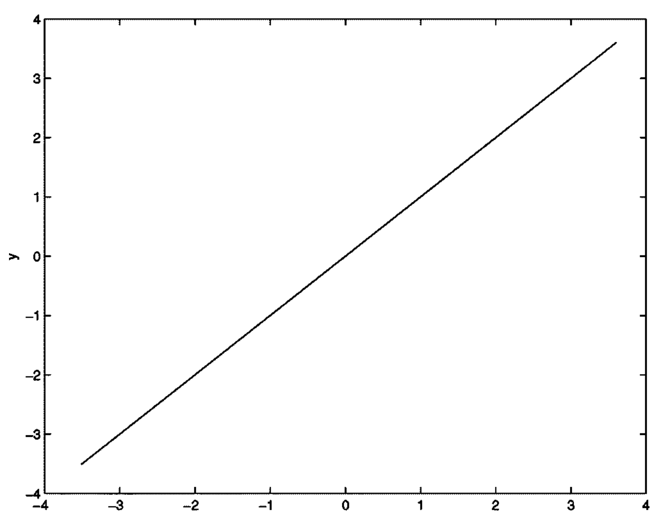

FIG. 3. Numerical simulation of systems (7) and (8): complete synchronization between y and x. The parameters are $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=1$, $m_1=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-17$. Dimensionless units.

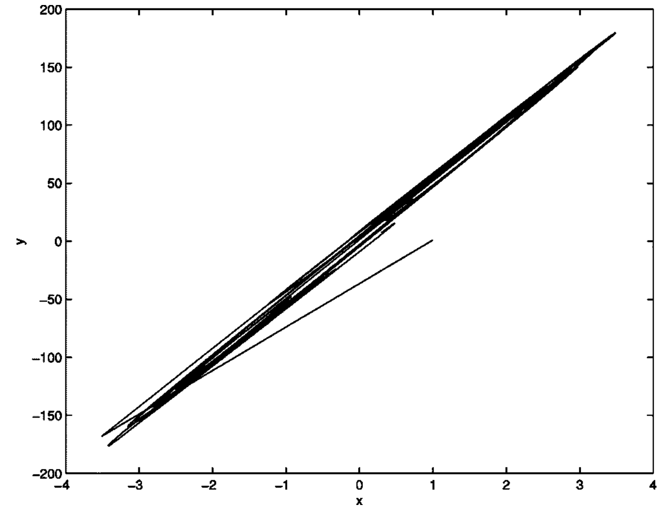

FIG. 4. Numerical simulation of systems (7) and (8): synchronization between y and x. The parameters are $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=1$, $m_1=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-1000$. Dimensionless units.

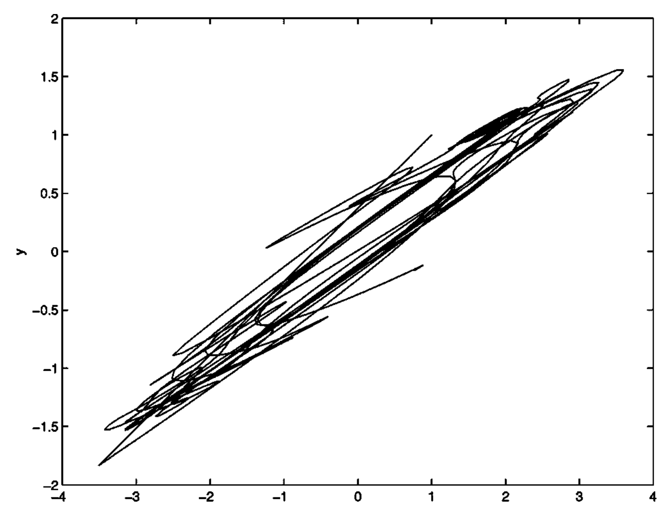

FIG. 5. Numerical simulation of systems (7) and (8): the dependence of y on x. The parameters are $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=1$, $m_1=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-8$. Dimensionless units.

$=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-17$. In Fig. 4 synchronization between x and y is shown for $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=1$, $m_1=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-1000$. We emphasize that as the coupling strength estimated from the stability condition gives a high-quality synchronization, the synchronization manifold is robust against perturbations of the coupling strength. But as mentioned above the onset of synchronization occurs at the coupling stength when the maximal Lyapunov exponent of the error dynamics vanishes as function of K. Our estimations show that for the parameters values as in Fig. 3 the threshold value of K is $K \approx -9.82$, which is (in absolute values) far less than $K=-17$. Figure 5 shows the dependence of y on x for $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=1$, $m_1=-20$, $m_3=-3$, $m_2=m_4=-1$, and $K=-8$.

IV. CONCLUSIONS

We have investigated different synchronization regimes between two unidirectionally coupled chaotic systems with multiple delays. We have found the necessary and sufficient stability conditions for the anticipating, lag, complete, and inverse synchronization manifolds. We have successfully applied the approach to the paradigm model in nonlinear physics—the Ikeda model. This research is of certain practical importance. It is well known that laser arrays hold great promise for space communication applications, which require compact sources with high optical intensities.The most efficient result can be achieved when the array elements are synchronized. Additional feedback mechanism could be useful to stabilize nonlinear system's output, e.g., laser intensity. Also having in mind different application possibilities of chaos synchronization, synchronization in multifeedback systems can provide more flexibility and opportunities in practical applications.

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